Enrollment No: ______________________ **Exam Seat No:______________________**

C.U.SHAH UNIVERSITY

Summer Examination-2016

Subject Name : Ring Theory

Subject Code : 4SC06RTC1 **Branch: B.Sc.(Mathematics)**

Semester : 6 **Date : 09/05/2016** **Time : 02:30 To 05:30** **Marks : 70**

Instructions:

(1) Use of Programmable calculator & any other electronic instrument is prohibited.
(2) Instructions written on main answer book are strictly to be obeyed.
(3) Draw neat diagrams and figures (if necessary) at right places.
(4) Assume suitable data if needed.

Q-1 **Attempt the following questions:** **(14)**

a) Show that an integral domain contains no idempotent except 0 or 1. **(02)**

b) Is $(\mathbb{Z}_{12}, +_{12}, \cdot_{12})$ an integral domain? Justify your answer. **(02)**

c) Show that $\mathbb{Z}$ is not an ideal of $\mathbb{Q}$. **(02)**

d) Show that the mapping $\phi: \left(\mathbb{Z}(\sqrt{2}), +, \cdot\right) \to \left(\mathbb{Z}(\sqrt{2}), +, \ \cdot\right)$, where $\phi(a) = m - n\sqrt{2}, a \in m + n\sqrt{2} \in \mathbb{Z}(\sqrt{2})$ is a ring homomorphism. **(02)**

e) Show that for nonzero polynomials f and $g \in D[x]$, $0 \leq [f] \leq [fg]$. **(02)**

f) Show that the polynomial $x^2 - 2$ has no zero in $\mathbb{Q}$. Is it has zero in $\mathbb{R}$? Justify. **(02)**

g) Is $\mathbb{Z}_{331}$ a field? Justify your answer. **(02)**

Attempt any four questions from Q-2 to Q-8

Q-2 **Attempt all questions** **(14)**

a) Obtain the g.c.d. of two polynomials $f(x) = 6x^3 + 5x^2 - 2x + 25$ and $g(x) = 2x^2 - 3x + 5 \in \mathbb{R}[x]$ and express it in the form $a(x)f(x) + b(x)g(x)$. **(06)**

b) Define characteristic of a ring. Show that if the characteristic of an integral domain D is p, then prove that $(a+b)^p = a^p + b^p, a, b \in D$. **(06)**

c) Show that $I = \left\{\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} / a, b \in \mathbb{Z}\right\}$ is not an right ideal of $M_2(\mathbb{Z})$. **(02)**

Q-3 **Attempt all questions** **(14)**

a) Define monic integer polynomial. Obtain all rational solutions of polynomial $f(x) = 4x^5 + x^3 + x^2 - 3x + 1$. **(06)**

b) Show that every field is an integral domain. Is converse true? Justify your answer. **(06)**

c) For $f(x), g(x), h(x) \in F[x]$ with $f(x) \ / \ g(x)h(x)$ and g.c.d. of $f(x)$ and $g(x)$ is 1, then show that $f(x)/h(x)$. **(02)**

Q-4		**Attempt all questions**	**(14)**
	a)	Check whether the set $\{a + b\sqrt{2} / a, b \in \mathbb{Z}\}$ is a ring under usual addition and multiplication or not.	**(06)**
	b)	If $r(I) = \{x \in R / x . u = 0; \forall u \in I\}$, where I is an ideal of ring R, then prove that $r(I)$ is an ideal of ring R.	**(06)**
	c)	Show that $f(x) = x^3 + x^2 + 1$ is irreducible over $\mathbb{Z}_2$.	**(02)**
Q-5		**Attempt all questions**	**(14)**
	a)	State and prove Eisenstein's criterion.	**(06)**
	b)	For any prime p, show that $(\mathbb{Z}_p, +_p, \cdot_p)$ is a field.	**(06)**
	c)	If $\phi: R \to R^{'}$ is a homomorphism then show that $(i) \phi(0) = 0^{'}$, $(ii) \phi(-a) = -\phi(a)$.	**(02)**
Q-6		**Attempt all questions**	**(14)**
	a)	If $\phi: R \to R^{'}$ is a homomorphism then prove kernel of ϕ is an ideal of R. Find kernel of ϕ, where $\phi: (\mathbb{Z}, +, \cdot) \to (\mathbb{Z}_2, +_2, \cdot_2)$ defined as $\phi(x) = \begin{cases} 0, x \text{ is even} \\ 1, x \text{ is odd} \end{cases}$.	**(06)**
	b)	Define field. If F is a field then show that F has only two ideals $\{0\}$ and F itself.	**(06)**
	c)	Check whether $g(x) = x + 1 \in \mathbb{R}[x]$ is a factor of $f(x) = 2x^3 + x + 3 \in \mathbb{R}[x]$ or not.	**(02)**
Q-7		**Attempt all questions**	**(14)**
	a)	If $\phi: R \to R^{'}$ is an onto homomorphism with kernel U then prove that $R / U \cong R^{'}$.	**(06)**
	b)	Show that the polynomial $f(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ is irreducible over $\mathbb{Q}$.	**(06)**
	c)	If I is an ideal of a ring R and $1 \in I$ then prove that $I = R$.	**(02)**
Q-8		**Attempt all questions**	**(14)**
	a)	State and prove division algorithm for polynomials.	**(06)**
	b)	Prove that characteristic of an integral domain D is either $'0'$ or prime.	**(06)**
	c)	Consider the ideal $I = 7\mathbb{Z}$ in the ring $(\mathbb{Z}, +, \cdot)$. If A is any ideal of $\mathbb{Z}$ with $I \subset A$ then either $A = I$ or $A = \mathbb{Z}$.	**(02)**